Available online at www.sciencedirect.com
ScienceDirect
Journal of Applied Mathematics and Mechanics 70 (2006) 195–198
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech

Invariant sets in the Goryachev–Chaplygin problem: existence, stability and branching☆

A.V. Karapetyan

Moscow

Received 23 June 2005

Abstract

The existence, stability and branching of invariant sets in the problem of the motion of a heavy rigid body with a fixed point, which satisfies the Goryachev–Chaplygin conditions, are discussed. Both trivial invariant sets, in which the pendulum-like motions of a Goryachev–Chaplygin spinning top lie, as well as non-trivial invariant sets, in which the motion of the top is described by elliptic functions of time, are investigated.

1. Formulation of the problem

Consider a heavy rigid body with a fixed point which obeys the Goryachev–Chaplygin conditions.[1,2]

Suppose P is the weight of the body, A, B and C ($A=B=4C$) are the principal moments of inertia of the body with respect to the fixed point, $x=a>0$, $y=z=0$ are the coordinates of the centre of mass in the corresponding axes and ω_1, ω_2, ω_3 and γ_1, γ_2, γ_3 are the projections of the angular velocity of the body and the projections of the unit vector of the upward vertical onto these axes.

Introducing the notation $\omega^2=Pa/C$ and assuming, without loss of generality, that $C=1$, we reduce the equations of motion of the body in the Euler-Poisson form to the form

$$4\dot{\omega}_1 = 3\omega_2\omega_3, \quad 4\dot{\omega}_2 = -3\omega_3\omega_1+\omega^2\gamma_3, \quad \dot{\omega}_3 = -\omega^2\gamma_2 \tag{1.1}$$

$$\dot{\gamma}_1 = \omega_3\gamma_2-\omega_2\gamma_3, \quad \dot{\gamma}_2 = \omega_1\gamma_3-\omega_3\gamma_1, \quad \dot{\gamma}_3 = \omega_2\gamma_1-\omega_1\gamma_2 \tag{1.2}$$

It is well known that Eqs. (1.1) and (1.2) allow of energy integral $H=\text{const}$, area integral $K=\text{const}$ and a geometric integral $\Gamma=1$ and, at the zeroth level of an area integral, the Goryachev–Chaplygin integral $G=\text{const}$:

$$H = \frac{1}{2}[4(\omega_1^2+\omega_2^2)+\omega_3^2]+\omega^2\gamma_1 = \omega^2 h \quad (h\in[-1,+\infty)) \tag{1.3}$$

$$K = 4(\omega_1\gamma_1+\omega_2\gamma_2)+\omega_3\gamma_3 = 0 \tag{1.4}$$

$$\Gamma = \gamma_1^2+\gamma_2^2+\gamma_3^2 = 1 \tag{1.5}$$

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 2, pp. 221–224, 2006.

0021-8928/$ – see front matter
doi:10.1016/j.jappmathmech.2006.06.003

$$G = (\omega_1^2 + \omega_2^2)\omega_3 - \omega^2\omega_1\gamma_3 = \omega^3 g \quad (g \in (-\infty, +\infty)) \tag{1.6}$$

According to the modified Routh theory,[3–7] the critical sets of one of the integrals (1.3) or (1.6) at fixed levels of all the remaining integrals ((1.4)–(1.6) or (1.3)–(1.5)) correspond to the invariant sets of system (1.1), (1.2). We shall seek the critical sets of the integral G on the fixed levels of the integrals $H = \omega^2 h$, $K = 0$ and $\Gamma = 1$. To do this, we introduce the function

$$W = G + \lambda(H - \omega^2 h) + \mu K + 1/2\nu(\Gamma - 1)$$

where λ, μ and ν are undetermined Lagrange multipliers and we now write out the conditions for it to be stationary with respect to the variables ω_1, ω_2, ω_3 and γ_1, γ_2, γ_3.

$$\begin{aligned}
\partial W/\partial\omega_1 &= 2\omega_1\omega_3 + 4\lambda\omega_1 + 4\mu\gamma_1 - \omega^2\gamma_3 = 0\\
\partial W/\partial\omega_2 &= 2\omega_2\omega_3 + 4\lambda\omega_2 + 4\mu\gamma_2 = 0\\
\partial W/\partial\omega_3 &= (\omega_1^2 + \omega_2^2) + \lambda\omega_3 + \mu\gamma_3 = 0\\
\partial W/\partial\gamma_1 &= \lambda\omega^2 + 4\mu\omega_1 + \nu\gamma_1 = 0, \quad \partial W/\partial\gamma_2 = 4\mu\omega_2 + \nu\gamma_2 = 0\\
\partial W/\partial\gamma_3 &= -\omega^2\omega_1 + \mu\omega_3 + \nu\gamma_3 = 0
\end{aligned} \tag{1.7}$$

It is now necessary to supplement Eqs. (1.7) with Eqs. (1.3)–(1.5), which are the conditions for the function W to be stationary with respect to the variables λ, μ and ν.

2. Trivial invariant sets

We will first consider a problem when all the undetermined Lagrange multipliers are equal to zero ($\lambda = \mu = \nu = 0$). In this case, it follows from Eqs. (1.3)–(1.5) and (1.7) that the six phase variables of system (1.1), (1.2) are constrained by the five relations

$$\omega_1 = \omega_2 = \gamma_3 = 0, \quad \gamma_1^2 + \gamma_2^2 = 1, \quad 1/2\omega_3^2 + \omega^2\gamma_1 = \omega^2 h \tag{2.1}$$

This means that relations (2.1) define families of one-dimensional invariant sets which are parametrized by the dimensionless constant h of the energy integral. At the same time, the body executes pendulum-like motions: oscillations when $h \in (-1, 1)$ and rotations when $h > 1$. When $h = -1$, the body finds itself in a stable equilibrium position and, when $h = 1$, it is either in an unstable equilibrium position or executes asymptotic motions when $t \to \pm\infty$. Actually, putting (see (2.1)) $\gamma_1 = \sin\varphi$, $\gamma_2 = \cos\varphi$, $\omega_3 = \dot\varphi$, we have (see Eqs. (1.1) and (1.2)) that $\varphi = \varphi(t)$ is determined from the equation

$$\ddot\varphi + \omega^2\cos\varphi = 0 \tag{2.2}$$

where

$$1/2\dot\varphi^2 + \omega^2\sin\varphi = \omega^2 h \tag{2.3}$$

Note that, when $h \in (-1, 1)$, a single family of invariant sets exists in which the oscillations of the body lie and, when $h > 1$, two families exist which correspond to the clockwise ($\dot\varphi < 0$) and counterclockwise ($\dot\varphi > 0$) rotations of the body during which, for any $h \in [-1, +\infty)$, the integral G takes a zero value ($g = 0$) in the invariant sets (2.1).

On calculating the second variation of the function W in the neighbourhood of the invariant sets (2.1) and determining the linear manifold $\delta H = \delta K = \delta\Gamma = 0$, we have

$$2\delta^2 W = 2\dot\varphi(\omega_1^2 + \omega_2^2) - 2\omega^2\omega_1\gamma_3 \tag{2.4}$$

$$\delta K = 4(\omega_1\sin\varphi + \omega_2\cos\varphi) + \dot\varphi\gamma_3 = 0 \tag{2.5}$$

The quadratic form (2.4) of the variables ω_1, ω_2 and γ_3 in the linear manifold (2.5) in the space of these variables is definite (indefinite) if the determinant

$$\Delta = -\begin{vmatrix} 0 & 4\sin\varphi & 4\cos\varphi & \dot{\varphi} \\ 4\sin\varphi & 2\dot{\varphi} & 0 & -\omega^2 \\ 4\cos\varphi & 0 & 2\dot{\varphi} & 0 \\ \dot{\varphi} & -\omega^2 & 0 & 0 \end{vmatrix}$$

is positive (negative). Here $\varphi = \varphi(t)$ is the solution of Eq. (2.2) for which relation (2.3) holds. Taking account of the latter fact, we have that $\Delta = 16\omega^4(h^2 - 1)$. Hence, the invariant sets (2.1) import a saddle value, when $h \in (-1, 1)$, and an extremum value, when $h \in (1, +\infty)$, to the integral (1.6) on the fixed levels of the integrals (1.3)–(1.5) (the extremum value is a minimum when $\dot{\varphi} > 0$ and a maximum when $\dot{\varphi} < 0$, since the principle diagonal third-order minor Δ_3 of the determinant Δ is equal to $\Delta_3 = 32\dot{\varphi}$). Consequently,[7] when $h \in (-1, 1)$, the invariant sets (2.1) are unstable and, when $h \in (1, +\infty)$, they are stable. These conclusions agree completely with the results in Ref. 8 in which the orbital instability of the oscillatory pendulum-like motions of a Goryachev–Chaplygin top and the orbital stability of the rotational pendulum-like motions of such a top was proved.

3. Non-trivial invariant sets

We will now consider the case when not all of the undetermined Lagrange multipliers are equal to zero ($\lambda^2+\mu^2+\nu^2 \neq 0$). Eliminating these multipliers from equations (1.7), we reduce these equations to the form (taking account of relation (1.5))

$$\begin{gathered} \omega_1\gamma_2 \mp \omega_2(1 \pm \gamma_1) = 0, \quad \omega_3\gamma_2\gamma_3 + 4\omega_2(1 \pm \gamma_1 - \gamma_3^2) = 0 \\ 4[3\gamma_3^2 - 2(1 \pm \gamma_1)]\omega_2^2 = \pm\omega^2\gamma_2^2\gamma_3^2 \end{gathered} \tag{3.1}$$

Here, relation (1.4) is satisfied identically and relation (1.3) takes the form

$$h \pm 1 = \pm\frac{3}{2}\frac{\gamma_3^4}{3\gamma_3^2 - 2(1 \pm \gamma_1)} \tag{3.2}$$

The five relations (1.5), (3.1) and (3.2), connecting the six phase variables of system (1.1), (1.2), define the two pairs of families of one-dimensional invariant sets of this system, which are parametrized by the dimensionless constant of the energy integral h. The first pair corresponds to the upper sign in relations (3.1) and (3.2) and, consequently, exists when $h \in [-1, +\infty)$ (here $3\gamma_3^2 > 2(1 + \gamma_1)$) and the second pair corresponds to the lower sign and, consequently, exists when $h \in [1, +\infty)$ (here $3\gamma_3^2 < 2(1 - \gamma_1)$). The families appearing in one or other of these pairs are distinguished by the direction of rotation (see relation (3.1)).

Calculating the dimensionless quantity g of the integral G in the invariant sets, determined by the combination of relations (1.5), (3.1) and (3.2), we have

$$27g^2 = 2(h \pm 1)^3 \tag{3.3}$$

(the upper sign (as previously) corresponds to the first pair of families and the lower sign to the second pair; the families occuring in one or other pair are distinguished by the sign of g). Note that a relation of the form of (3.3) was previously obtained from other considerations in Ref. 9.

Recalling that $g = 0$ in the invariant sets (2.1), we construct a Poincaré- Smale bifurcation diagram (see Fig. 1) in the (h, g)-plane. The line $g = 0$ corresponds to the family of trivial invariant sets and curves *1* and *2* correspond to the first and second pairs of families of non-trivial invariant sets. The stable and unstable invariant sets (the stability of non-trivial sets is defined in accordance with bifurcation theory) are labelled with plus and minus signs respectively.

In conclusion, we note that the motion of a Goryachev–Chaplygin top on the non-trivial invariant sets is described by elliptic functions of time. Actually, it follows from Eqs. (1.1), and (1.2), when account is taken of relations (3.1)

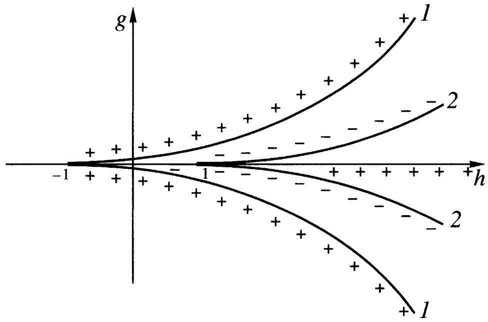

Fig. 1.

and (3.2), that

$$\dot{\gamma}_3 = \omega_2; \quad \omega_2^2 = \frac{\omega^2}{96(h\pm 1)}F_\pm(\gamma_3)$$
$$F_\pm(\gamma_3) = 32(h\pm 1)^2 - 12(h\pm 1)(3h\pm 5)\gamma_3^2 \pm 36(h\pm 1)\gamma_3^4 - 9\gamma_3^6 \tag{3.4}$$

Knowing $\gamma_3(t)$ (see Eq. (3.4)), it is possible to find the function $\omega_2(t)$ (see (3.4)), $\gamma_1(t)$ (see (3.2)) and $\gamma_2(t)$ (see (3.2) and (1.5)), and also $\omega_1(t)$ and $\omega_3(t)$ (see the first two equalities of (3.1)). In particular,

$$\gamma_1 = \mp\left[1 - \frac{3}{2}\gamma_3^2 \pm \frac{3}{4}\frac{\gamma_3^4}{(h\pm 1)}\right], \quad \gamma_2^2 = \frac{\gamma_3^2}{16(h\pm 1)^2}F_\pm(\gamma_3) \tag{3.5}$$

Note that relations (3.5) determine the non-trivial invariant sets on the Poisson sphere (1.5) (we recall that the corresponding trivial sets have the form $\gamma_1^2 + \gamma_2^2 = 1$, $\gamma_3 = 0$).

Acknowledgement

This research was supported financially by the Russian Foundation for Basic Research (04-01-00398) and the "Universities of Russia" programme.

References

1. Goryachev DN. The motion of a heavy rigid body about a fixed point in the case when $A = B = 4C$. *Mat Sbornik Kruzhka Lyubitelei Mat Nauk* 1900;**21**(3):431–8.
2. Chaplygin SA. A new particular solution of the problem of the rotation of a heavy rigid body about a fixed point. *Trudy Otd Fiz Nauk Obshch Lyubitelei Yestestvoznaniya* 1904;**12**(1):1–4.
3. Routh EJ. *A Treatise of Stability of a Given State of Motion*. London: Macmillan; 1877, p 108.
4. Levi-Civita T. Sur la recherche des solution particulières des systèmes differentiels et sur les mouvment stationaires. *Prac mat -fis* 1906;**17**:1–40.
5. Salvadori L. Un osservazione su di un criterio di stabilità del Routh. *Reud Accad Sci Fis Math Napoli Ser 4* 1953;**20**(1–2):269–72.
6. Pozharitskii GK. The construction of Lyapunov functions from integrals of the equations of perturbed motion. *Prikl Mat Mekh* 1958;**22**(2):145–54.
7. Karapetyan AV. *The Stability of Steady Motions*. Moscow: Editorial URSS; 1998.
8. Markeyev AP. The pendulum-like motions of a rigid body in the Goryachev–Chaplygin case. *Prikl Mat Mekh* 2004;**68**(2):282–93.
9. Kharlamov MP. *Topological Analysis of Integrable Problems of Rigid Body Dynamics*. Leningrad: LGU; 1988.

Translated by E.L.S.